Generalized quantum Langevin equations from the forward-backward path integral

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Kleinert and Shabanov [H. Kleinert and S. V. Shabanov, Phys. Lett. A **200**, 224 (1995)] have derived the quantum Langevin equations from the Feynman-Vernon forward-backward path integral for a density matrix of a quantum system in a thermal oscillator bath. However, their derivation is confined to an Ohmic case. In this paper we derive the generalized quantum Langevin equations from the forward-backward path integral, by extending the Kleinert-Shabanov method to a general case. [S1063-651X(99)05005-9]

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I. INTRODUCTION

The influence functional path integral method of Feynman and Vernon [1,2] is very useful to study the behavior of a quantum system coupled with a heat bath. The state of the system is described by the reduced density matrix, derived by tracing out the environment coordinates in the full density matrix. The influence functional method provides an easy way to obtain a functional representation for the propagator for the reduced density matrix.

In recent years, some efforts have been made to derive the quasiclassical and quantum Langevin equations for the system from the Feynman-Vernon path integral [3–9]. In this paper, we are interested in deriving the quantum Langevin equations. For the system in a thermal oscillator bath, Kleinert and Shabanov have derived the quantum Langevin equations for an Ohmic environment [9]. In the derivation of the quantum Langevin equations, they make use of a Schrödinger-like differential equation for a noisy density matrix. The noisy density matrix is defined by replacing the real part of the exponent in the influence functional of the reduced density matrix, written as a double time integral, by a single time integral whose integrand is expressed with a quantum noise operator. The Schrödinger-like differential equation can be derived by analogy with the procedure to obtain the Schrödinger equation from the path integral formulation. The analogous procedure to obtain the Schrödinger-like differential equation can exist only when the effective action in the propagator for the noisy density matrix is given by a single time integral. In an Ohmic case, there is the analogous procedure since the imaginary part of the exponent in the influence functional also becomes a single time integral. However, in a non-Ohmic case, there is no analogous procedure due to a double time integral in the imaginary part of the exponent in the influence functional. Therefore, the Kleinert-Shabanov method has difficulty in deriving the generalized quantum Langevin equations for a general environment. The purpose of this paper is to overcome the difficulty and derive the generalized quantum Langevin equations for the general case.

In Sec. II we explain the Feynman-Vernon forward-backward path integral for the reduced density matrix of a system in a general environment. Then in Sec. III we introduce the noisy density matrix for the general environment according to the Kleinert-Shabanov method for the Ohmic case. In Sec. IV we summarize the derivation of the quantum Langevin equations for an Ohmic environment. In Sec. V we show how to derive the generalized quantum Langevin equations for a general environment. Section VI contains some discussions.

II. FEYNMAN-VERNON FORWARD-BACKWARD PATH INTEGRAL FOR THE REDUCED DENSITY MATRIX

We consider a quantum system with mass m moving in a potential $V(x)$ and bilinearly coupled to a thermal oscillator bath consisting of a set of harmonic oscillators with mass m_n and natural frequency ω_n. The Hamiltonian of the system plus environment is given by

$$\hat{H}=\frac{\hat{p}^2}{2m}+V(\hat{x})+\sum_{n=1}^{N}\left[\frac{\hat{p}_n^2}{2m_n}+\frac{1}{2}m_n\omega_n^2\left(\hat{q}_n-\frac{c_n}{m_n\omega_n^2}\hat{x}\right)^2\right], \tag{1}$$

where $\hat{x}$ and $\hat{p}$ are the coordinate and momentum operators of the system, $\hat{q}_n$ and $\hat{p}_n$ are those of the oscillators, and c_n are coupling constants. Here we have the canonical commutation relations

$$[\hat{x},\hat{p}]=i\hbar, \quad [\hat{q}_n,\hat{p}_m]=i\hbar\,\delta_{nm}, \tag{2}$$

and all other commutators vanish.

Let $\hat{\rho}(t)$ be the density matrix for the system plus environment. It evolves according to

$$\dot{\hat{\rho}}(t)=-\frac{i}{\hbar}[\hat{H},\hat{\rho}(t)] \tag{3}$$

and reads

$$\hat{\rho}(t)=\exp\left(-\frac{i\hat{H}t}{\hbar}\right)\hat{\rho}(0)\left(\frac{i\hat{H}t}{\hbar}\right). \tag{4}$$

The state of the system is described by the reduced density matrix, defined as

$$\rho^S(x,x',t)=\langle x|\hat\rho^S(t)|x'\rangle = \int_{-\infty}^{+\infty}dq\int_{-\infty}^{+\infty}dq'\langle xq|\hat\rho(t)|x'q'\rangle\delta(q-q'). \tag{5}$$

Here we have used q to denote the full set of oscillator coordinates q_n. If we assume that at some initial time $t=0$ the system and the bath are decoupled and the bath is in thermal equilibrium at temperature $T=(k_B\beta)^{-1}$, then we can write Eq. (5) as [1,2,7,8]

$$\rho^S(x,x',t) = \int_{-\infty}^{+\infty}dx_i\int_{-\infty}^{+\infty}dx_i'\,U(x,x',t;x_i,x_i',0)\rho^S(x_i,x_i',0), \tag{6}$$

where

$$U(x,x',t;x_i,x_i',0) = \int_{x_i}^{x}Dx_+\int_{x_i'}^{x'}Dx_-F[x_+,x_-] \times\exp\left[\frac{i}{\hbar}(A^S[x_+]+A^C[x_+]-A^S[x_-]-A^C[x_-])\right] \equiv\int_{x_i}^{x}Dx_+\int_{x_i'}^{x'}Dx_-\exp\left(\frac{i}{\hbar}A[x_+,x_-]\right). \tag{7}$$

Here the subscript i denotes initial variables, the functional $A[x_+,x_-]$ is the effective action for the system, and the functionals $A^S[x_\pm]$, $A^C[x_\pm]$ are the classical actions associated with the system Hamiltonian and the term $\hat x^2\Sigma_{n=1}^N c_n^2/2m_n\omega_n^2$ in Eq. (1), respectively,

$$A^S[x_\pm]=\int_0^t ds\left[\frac{m}{2}\dot x_\pm(s)^2-V(x_\pm(s))\right], \tag{8}$$

$$A^C[x_\pm]=-\int_0^t ds f(0)x_\pm(s)^2,$$

where

$$f(s)=\int_0^{+\infty}d\omega\frac{I(\omega)}{\omega}\cos(\omega s), \tag{9}$$

where $I(\omega)$ is the spectral density of the environment:

$$I(\omega)=\sum_{n=1}^N\delta(\omega-\omega_n)\frac{c_n^2}{2m_n\omega_n}. \tag{10}$$

Here it is supposed that the frequencies of the oscillators are distributed along the positive real axis. The influence functional $F[x_+,x_-]$ can be written as

$$F[x_+,x_-]=F^I[X,Y]F^R[Y], \tag{11}$$

$$F^I[X,Y]=\exp\left[-\frac{2i}{\hbar}\int_0^t ds\int_0^s du\,Y(s)\alpha^I(s-u)X(u)\right], \tag{12}$$

$$F^R[Y]=\exp\left[-\frac{1}{\hbar}\int_0^t ds\int_0^s du\,Y(s)\alpha^R(s-u)Y(u)\right], \tag{13}$$

where we have introduced new variables

$$X(s)=[x_+(s)+x_-(s)]/2,\quad Y(s)=x_+(s)-x_-(s). \tag{14}$$

The kernels $\alpha^R(s)$ and $\alpha^I(s)$ are defined as

$$\alpha^R(s)=\int_0^{+\infty}d\omega I(\omega)\coth\left(\frac{1}{2}\beta\hbar\omega\right)\cos(\omega s) \tag{15}$$

and

$$\alpha^I(s)=\frac{d}{ds}f(s). \tag{16}$$

Since the paths $x_+(s)$, $x_-(s)$ correspond to a forward and backward movement of the system in time, the expression (7) is also called forward-backward path integral.

III. NOISY DENSITY MATRIX

Let us consider the quantum noise operator

$$\hat\eta(s)=\sum_{n=1}^N c_n\left[\hat q_n\cos(\omega_n s)+\frac{\hat p_n}{m_n}\frac{\sin(\omega_n s)}{\omega_n}\right]. \tag{17}$$

This satisfies the commutation relation

$$[\hat\eta(s),\hat\eta(s')]=2i\hbar\alpha^I(s-s'). \tag{18}$$

We shall demonstrate that the double time integral in Eq. (13) can be replaced by a single time integral with the help of this noise operator. To this end, we first introduce the adjoining operator $\hat\eta_c(s)$ associated with $\hat\eta(s)$. This is defined as

$$\hat\eta_c(s)\hat O=\frac{1}{2}[\hat\eta(s),\hat O]_+, \tag{19}$$

where $\hat O$ is an arbitrary operator. For convenience we have adopted the notation for the adjoining operator used in Refs. [10] and [11]. Although the noise operator $\hat\eta(s)$ has a nontrivial commutator (18) with itself at a different time, two adjoining operators $\hat\eta_c(s)$ and $\hat\eta_c(s')$ commute with each other, so that they can be treated as c numbers:

$$[\hat\eta_c(s),\hat\eta_c(s')]\hat O=\frac{1}{4}[\hat\eta(s),\hat\eta(s')]\hat O +\frac{1}{4}\hat O[\hat\eta(s'),\hat\eta(s)]=0. \tag{20}$$

We next consider the correlation function of $\hat\eta_c(s)$,

$$\langle \hat{\eta}_c(s)\hat{\eta}_c(s')\rangle \equiv \mathrm{Tr}_B[\hat{\eta}_c(s)\hat{\eta}_c(s')\hat{\rho}^B], \tag{21}$$

where Tr_B denotes the trace over bath degrees of freedom and $\hat{\rho}^B$ is the initial equilibrium density operator for the bath:

$$\hat{\rho}^B = \exp(-\beta \hat{H}^B)/\mathrm{Tr}_B[\exp(-\beta \hat{H}^B)], \tag{22}$$

where $\hat{H}^B$ is the bath Hamiltonian:

$$\hat{H}^B = \sum_{n=1}^{N}\left[\frac{\hat{p}_n^2}{2m_n} + \frac{1}{2}m_n\omega_n^2\hat{q}_n^2\right]. \tag{23}$$

This is calculated as

$$\langle \hat{\eta}_c(s)\hat{\eta}_c(s')\rangle = \frac{1}{2}\langle[\hat{\eta}(s),\hat{\eta}(s')]_+\rangle = \hbar\alpha^R(s-s'). \tag{24}$$

Note that the result agrees with the kernel of Eq. (13) within the factor $\hbar$. Since the adjoining operator $\hat{\eta}_c(s)$ is a Gaussian noise variable, we may rewrite Eq. (13) as [8,9]

$$F^R[Y] = \left\langle \exp\left[\frac{i}{\hbar}\int_0^t ds Y(s)\hat{\eta}_c(s)\right]\right\rangle. \tag{25}$$

Then we can replace the double time integral in Eq. (13) by the single time integral in Eq. (25), by introducing the noisy density matrix of the system

$$\rho^S_{\hat{\eta}_c}(x,x',t) = \langle x|\hat{\rho}^S_{\hat{\eta}_c}(t)|x'\rangle = \int_{-\infty}^{+\infty}dx_i\int_{-\infty}^{+\infty}dx_i' \times U_{\hat{\eta}_c}(x,x',t;x_i,x_i',0)\rho^S(x_i,x_i',0), \tag{26}$$

where

$$U_{\hat{\eta}_c}(x,x',t;x_i,x_i',0) = \int_{x_i}^{x}Dx_+\int_{x_i'}^{x'}Dx_- F_{\hat{\eta}_c}[x_+,x_-]\exp\left[\frac{i}{\hbar}(A^S[x_+] + A^C[x_+] - A^S[x_-] - A^C[x_-])\right], \tag{27}$$

$$F_{\hat{\eta}_c}[x_+,x_-] = F^I[X,Y]F^R_{\hat{\eta}_c}[Y], \tag{28}$$

$$F^R_{\hat{\eta}_c}[Y] = \exp\left[\frac{i}{\hbar}\int_0^t ds Y(s)\hat{\eta}_c(s)\right]. \tag{29}$$

Then the reduced density matrix is given by the bath average of the noisy density matrix:

$$\langle \hat{\rho}^S_{\hat{\eta}_c}(t)\rangle = \hat{\rho}^S(t). \tag{30}$$

Using the variables $X(s)$, $Y(s)$ defined by Eq. (14), the noise-dependent effective action $A_{\hat{\eta}_c}[X,Y]$ in the propagator (27) is written as

$$A_{\hat{\eta}_c}[X,Y] = \int_0^t ds\{m\dot{Y}(s)\dot{X}(s) - V[X(s)+Y(s)/2] + V[X(s) - Y(s)/2] - 2f(0)Y(s)X(s)\} - 2\int_0^t ds\int_0^s du Y(s)\alpha^I(s-u)X(u) + \int_0^t ds Y(s)\hat{\eta}_c(s). \tag{31}$$

IV. DERIVATION OF THE QUANTUM LANGEVIN EQUATIONS

Now we consider the Ohmic case

$$I(\omega) = \frac{\gamma}{\pi}\omega. \tag{32}$$

Substituting Eq. (32) into Eq. (12) and integrating by parts with respect to u, we obtain

$$F^I[X,Y] = \exp\left[-\frac{i\gamma}{\hbar}\int_0^t ds Y(s)\dot{X}(s) - \frac{2i\gamma}{\hbar}\int_0^t ds Y(s)\delta(s)X(s) + \frac{2i\gamma}{\hbar}\int_0^t ds Y(s)\delta(0)X(s)\right]. \tag{33}$$

The term $\int_0^t ds[-2f(0)Y(s)X(s)]$ in Eq. (31) is canceled by the last term of the exponent in Eq. (33). Then Eq. (31) becomes

$$A_{\hat{\eta}_c}[X,Y] = \int_0^t ds L_{\hat{\eta}_c}(X(s),Y(s),\dot{X}(s),\dot{Y}(s)), \tag{34}$$

where

$$L_{\hat{\eta}_c}(X(s),Y(s),\dot{X}(s),\dot{Y}(s)) = m\dot{Y}(s)\dot{X}(s) - V[X(s) + Y(s)/2] + V[X(s)-Y(s)/2] - \gamma Y(s)\dot{X}(s) + Y(s)\hat{\eta}_c(s) - 2\gamma Y(s)\delta(s)X(s). \tag{35}$$

The action (34) can be rewritten in the canonical form:

$$A_{\hat{\eta}_c}[P_X,P_Y,X,Y] = \int_0^t ds[P_X(s)\dot{X}(s) + P_Y(s)\dot{Y}(s) - H_{\hat{\eta}_c}(P_X(s),P_Y(s),X(s),Y(s))], \tag{36}$$

where

$$H_{\hat{\eta}_c}(P_X(s),P_Y(s),X(s),Y(s))$$
$$=\frac{1}{m}[P_X(s)+\gamma Y(s)]P_Y(s)+V[X(s)+Y(s)/2]$$
$$-V[X(s)-Y(s)/2]-Y(s)\hat{\eta}_c(s)+2\gamma Y(s)\delta(s)X(s). \tag{37}$$

Here $P_X(s)$, $P_Y(s)$ are the generalized momenta:

$$P_X(s)=\frac{\partial}{\partial\dot{X}(s)}L_{\hat{\eta}_c}(X(s),Y(s),\dot{X}(s),\dot{Y}(s)),$$
$$P_Y(s)=\frac{\partial}{\partial\dot{Y}(s)}L_{\hat{\eta}_c}(X(s),Y(s),\dot{X}(s),\dot{Y}(s)). \tag{38}$$

Since the action (36) is the single time integral, we obtain the Schrödinger-like differential equation for the noisy density matrix (26),

$$i\hbar\frac{\partial}{\partial t}\rho^S_{\hat{\eta}_c}(X,Y,t)=\hat{H}_{\hat{\eta}_c}\rho^S_{\hat{\eta}_c}(X,Y,t), \tag{39}$$

with $\hat{H}_{\hat{\eta}_c}$ being the operator arising from Eq. (37) by substituting $-i\hbar\partial/\partial X$ and $-i\hbar\partial/\partial Y$ into $P_X(t)$ and $P_Y(t)$, respectively. Here $X=X(t)=(x+x')/2$ and $Y=Y(t)=x-x'$. In the transition from the term $\gamma Y(t)P_Y(t)/m$ in Eq. (37) to the operator, we choose the operator ordering to be $\gamma Y(-i\hbar\partial/\partial Y)/m$, not $\gamma(-i\hbar\partial/\partial Y)Y/m$ [9]. Then Eq. (39) can be rewritten as

$$i\hbar\frac{\partial}{\partial t}\hat{\rho}^S_{\hat{\eta}_c}(t)=[\hat{H}^S,\hat{\rho}^S_{\hat{\eta}_c}(t)]+\frac{1}{2}[\hat{x},[\gamma\hat{p}/m-\hat{\eta}(t)$$
$$+2\gamma\delta(t)\hat{x},\hat{\rho}^S_{\hat{\eta}_c}(t)]_+], \tag{40}$$

where $\hat{H}^S$ is the system Hamiltonian:

$$\hat{H}^S=\frac{\hat{p}^2}{2m}+V(\hat{x}). \tag{41}$$

Let us define the noisy Heisenberg system operator $\hat{O}^S_{\hat{\eta}_c}(t)$ for the Schrödinger system operator $\hat{O}^S$ by

$$\mathrm{Tr}_S[\hat{O}^S\hat{\rho}^S_{\hat{\eta}_c}(t)]=\mathrm{Tr}_S[\hat{O}^S_{\hat{\eta}_c}(t)\hat{\rho}^S(0)], \tag{42}$$

where Tr_S implies a trace over system degrees of freedom. Since $\hat{\rho}^S_{\hat{\eta}_c}(0)=\hat{\rho}^S(0)$, the noisy Heisenberg system operator $\hat{O}^S_{\hat{\eta}_c}(t)$ coincides with the Schrödinger system operator $\hat{O}^S$ at $t=0$. Differentiating both sides of Eq. (42) with respect to t and using Eqs. (40) and (42), we get

$$\dot{\hat{x}}_{\hat{\eta}_c}(t)=\hat{p}_{\hat{\eta}_c}(t)/m,$$
$$\dot{\hat{p}}_{\hat{\eta}_c}(t)=-\hat{V}'_{\hat{\eta}_c}(t)-\gamma\dot{\hat{x}}_{\hat{\eta}_c}(t)+\hat{\eta}(t)-2\gamma\delta(t)\hat{x}, \tag{43}$$

where $\hat{V}'_{\hat{\eta}_c}(t)$ is the noisy Heisenberg system operator for $V'(\hat{x})$, and the prime denotes the derivative with respect to $\hat{x}$. If we recognize that $\hat{V}'_{\hat{\eta}_c}(t)=V'(\hat{x}_{\hat{\eta}_c}(t))$ [9], we obtain the same forms as the quantum Langevin equations derived from the Heisenberg equations of motion [12–16]. Then we can write

$$\hat{x}_{\hat{\eta}_c}(t)=e^{i\hat{H}t/\hbar}\hat{x}e^{-i\hat{H}t/\hbar},$$
$$\hat{p}_{\hat{\eta}_c}(t)=e^{i\hat{H}t/\hbar}\hat{p}e^{-i\hat{H}t/\hbar}. \tag{44}$$

V. DERIVATION OF THE GENERALIZED QUANTUM LANGEVIN EQUATIONS

Now we turn to the general case with the general spectral density defined in Eq. (10). Then the noise-dependent effective action $A_{\hat{\eta}_c}[X,Y]$ for the system is given by Eq. (31). Except for the Ohmic case, there is no analogous procedure to obtain the Schrödinger-like differential equation for the noisy density matrix (26), due to the double time integral in Eq. (31). Therefore, we meet the difficulty in deriving the generalized quantum Langevin equation for the general environment. To overcome this difficulty, we introduce two functions $W(s)$, $Z(s)$ and a functional $P[W,Z]$ and suppose the following path integral Fourier transform:

$$\exp\left[-\frac{2i}{\hbar}\int ds\int^s du Y(s)\alpha^I(s-u)X(u)\right]$$
$$=\int DW\int DZ P[W,Z]\exp\left(\frac{i}{\hbar}\int ds[X(s)W(s)+Y(s)Z(s)]\right)$$
$$\equiv\left\langle\exp\left(\frac{i}{\hbar}\int ds[X(s)W(s)+Y(s)Z(s)]\right)\right\rangle_{W,Z}. \tag{45}$$

That is to say, we suppose that the left-hand side of Eq. (45) is given by the average over all $W(s)$, $Z(s)$ with weight $P[W,Z]DWDZ$ of

$$\exp\left(\frac{i}{\hbar}\int ds[X(s)W(s)+Y(s)Z(s)]\right). \tag{46}$$

From Eq. (45) it is clear that

$$\langle 1\rangle_{W,Z}=\int DW\int DZ P[W,Z]=1. \tag{47}$$

By using Eq. (45) and setting $X(s)=Y(s)=0$ for $s>t$ and $s<0$, we obtain

$$F^I[X,Y]=\left\langle\exp\left(\frac{i}{\hbar}\int_0^t ds[X(s)W(s)+Y(s)Z(s)]\right)\right\rangle_{W,Z}. \tag{48}$$

Let us define the new functional $F^I_{W,Z}[X,Y]$ by

$$F^I_{W,Z}[X,Y]=\exp\left(\frac{i}{\hbar}\int_0^t ds[X(s)W(s)+Y(s)Z(s)]\right). \tag{49}$$

Using this functional, we introduce the new noisy density matrix of the system

$$\rho^S_{\hat\eta_c;W,Z}(x,x',t)=\langle x|\hat\rho^S_{\hat\eta_c;W,Z}(t)|x'\rangle =\int_{-\infty}^{+\infty}dx_i\int_{-\infty}^{+\infty}dx_i' U_{\hat\eta_c;W,Z}(x,x',t;x_i,x_i',0) \times\rho^S(x_i,x_i',0), \tag{50}$$

where

$$U_{\hat\eta_c;W,Z}(x,x',t;x_i,x_i',0) =\int_{x_i}^{x}Dx_+\int_{x_i'}^{x'}Dx_- F_{\hat\eta_c;W,Z}[x_+,x_-]\exp\left[\frac{i}{\hbar}(A^S[x_+] +A^C[x_+]-A^S[x_-]-A^C[x_-])\right], \tag{51}$$

$$F_{\hat\eta_c;W,Z}[x_+,x_-]=F^I_{W,Z}[X,Y]F^R_{\hat\eta_c}[Y]. \tag{52}$$

Then $\rho^S_{\hat\eta_c}(x,x',t)$ is given by the average of the new noisy density matrix (50) over all $W(s)$, $Z(s)$ with weight $P[W,Z]DWDZ$:

$$\langle\rho^S_{\hat\eta_c;W,Z}(x,x',t)\rangle_{W,Z}=\rho^S_{\hat\eta_c}(x,x',t). \tag{53}$$

Using the variables $X(s)$, $Y(s)$ defined by Eq. (14), the new noise-dependent effective action $A_{\hat\eta_c;W,Z}[X,Y]$ in the propagator (51) is written as

$$A_{\hat\eta_c;W,Z}[X,Y]=\int_0^t ds L_{\hat\eta_c;W,Z}(X(s),Y(s),\dot X(s),\dot Y(s)), \tag{54}$$

where

$$L_{\hat\eta_c;W,Z}(X(s),Y(s),\dot X(s),\dot Y(s)) =m\dot Y(s)\dot X(s)-V[X(s) +Y(s)/2]+V[X(s)-Y(s)/2] -2f(0)Y(s)X(s) +X(s)W(s)+Y(s)Z(s) +Y(s)\hat\eta_c(s). \tag{55}$$

The new action (54) can be rewritten in the canonical form:

$$A_{\hat\eta_c;W,Z}[P_X,P_Y,X,Y] =\int_0^t ds[P_X(s)\dot X(s)+P_Y(s)\dot Y(s) -H_{\hat\eta_c;W,Z}(P_X(s),P_Y(s),X(s),Y(s))], \tag{56}$$

where

$$H_{\hat\eta_c;W,Z}(P_X(s),P_Y(s),X(s),Y(s)) =\frac{1}{m}P_X(s)P_Y(s)+V[X(s)+Y(s)/2]-V[X(s) -Y(s)/2]+2f(0)Y(s)X(s)-X(s)W(s) -Y(s)Z(s)-Y(s)\hat\eta_c(s). \tag{57}$$

Here the generalized momenta $P_X(s)$, $P_Y(s)$ are defined by

$$P_X(s)=\frac{\partial}{\partial\dot X(s)}L_{\hat\eta_c;W,Z}(X(s),Y(s),\dot X(s),\dot Y(s)),$$
$$P_Y(s)=\frac{\partial}{\partial\dot Y(s)}L_{\hat\eta_c;W,Z}(X(s),Y(s),\dot X(s),\dot Y(s)). \tag{58}$$

Since the action (56) is the single time integral, we obtain the Schrödinger-like differential equation for the new noisy density matrix (50),

$$i\hbar\frac{\partial}{\partial t}\rho^S_{\hat\eta_c;W,Z}(X,Y,t)=\hat H_{\hat\eta_c;W,Z}\rho^S_{\hat\eta_c;W,Z}(X,Y,t), \tag{59}$$

with $\hat H_{\hat\eta_c;W,Z}$ being the operator arising from Eq. (57) by substituting $-i\hbar\partial/\partial X$ and $-i\hbar\partial/\partial Y$ into $P_X(t)$ and $P_Y(t)$, respectively. There exists no operator ordering problem as in Eq. (39). Averaging both sides of Eq. (59) over all $W(s)$, $Z(s)$ with the weight $P[W,Z]DWDZ$ and using Eq. (53), we obtain

$$i\hbar\frac{\partial}{\partial t}\rho^S_{\hat\eta_c}(X,Y,t)=\hat H'_{\hat\eta_c}\rho^S_{\hat\eta_c}(X,Y,t) -\langle XW(t)\rho^S_{\hat\eta_c;W,Z}(X,Y,t)\rangle_{W,Z} -\langle YZ(t)\rho^S_{\hat\eta_c;W,Z}(X,Y,t)\rangle_{W,Z}, \tag{60}$$

with $\hat H'_{\hat\eta_c}$ being the operator arising from

$$H'_{\hat\eta_c}(P_X(s),P_Y(s),X(s),Y(s)) =\frac{1}{m}P_X(s)P_Y(s)+V[X(s)+Y(s)/2]-V[X(s) -Y(s)/2]+2f(0)Y(s)X(s)-Y(s)\hat\eta_c(s), \tag{61}$$

by substituting $-i\hbar\partial/\partial X$ and $-i\hbar\partial/\partial Y$ into $P_X(t)$ and $P_Y(t)$, respectively. Here it should be noted that the weight $P[W,Z]DWDZ$ is independent of t. The last two terms on the right-hand side of Eq. (60) can be rewritten as follows:

$$
\begin{aligned}
\langle XW(t)\rho^S_{\hat\eta_c;W,Z}(X,Y,t)\rangle_{W,Z}
&= X\int_{-\infty}^{+\infty}dX_i\int_{-\infty}^{+\infty}dY_i\int_{X_i}^{X}DX\int_{Y_i}^{Y}DY \\
&\quad\times\left\langle \frac{2\hbar}{i}\frac{\delta}{\delta X(t)}(F^I_{W,Z}[X,Y])\right\rangle_{W,Z}F^R_{\hat\eta_c}[Y] \\
&\quad\times\exp\left(\frac{i}{\hbar}A^{S+C}[X,Y]\right)\rho^S(X_i,Y_i,0) \\
&= X\int_{-\infty}^{+\infty}dX_i\int_{-\infty}^{+\infty}dY_i\int_{X_i}^{X}DX\int_{Y_i}^{Y}DY \\
&\quad\times\frac{2\hbar}{i}\frac{\delta}{\delta X(t)}(\langle F^I_{W,Z}[X,Y]\rangle_{W,Z})F^R_{\hat\eta_c}[Y] \\
&\quad\times\exp\left(\frac{i}{\hbar}A^{S+C}[X,Y]\right)\rho^S(X_i,Y_i,0), \qquad (62)
\end{aligned}
$$

$$
\begin{aligned}
\langle YZ(t)\rho^S_{\hat\eta_c;W,Z}(X,Y,t)\rangle_{W,Z}
&= Y\int_{-\infty}^{+\infty}dX_i\int_{-\infty}^{+\infty}dY_i\int_{X_i}^{X}DX\int_{Y_i}^{Y}DY \\
&\quad\times\left\langle \frac{2\hbar}{i}\frac{\delta}{\delta Y(t)}(F^I_{W,Z}[X,Y])\right\rangle_{W,Z} \\
&\quad\times F^R_{\hat\eta_c}[Y]\exp\left(\frac{i}{\hbar}A^{S+C}[X,Y]\right)\rho^S(X_i,Y_i,0) \\
&= Y\int_{-\infty}^{+\infty}dX_i\int_{-\infty}^{+\infty}dY_i\int_{X_i}^{X}DX\int_{Y_i}^{Y}DY\,\frac{2\hbar}{i}\frac{\delta}{\delta Y(t)} \\
&\quad\times(\langle F^I_{W,Z}[X,Y]\rangle_{W,Z})F^R_{\hat\eta_c}[Y] \\
&\quad\times\exp\left(\frac{i}{\hbar}A^{S+C}[X,Y]\right)\rho^S(X_i,Y_i,0), \qquad (63)
\end{aligned}
$$

where $X_i=X(0)=(x_i+x_i')/2$, $Y_i=Y(0)=x_i-x_i'$, and $A^{S+C}[X,Y]=A^S[x_+]+A^C[x_+]-A^S[x_-]-A^C[x_-]$. From Eqs. (48), (49), and (12), it can be easily shown that

$$
\frac{2\hbar}{i}\frac{\delta}{\delta X(t)}(\langle F^I_{W,Z}[X,Y]\rangle_{W,Z})=0 \qquad (64)
$$

and

$$
\frac{2\hbar}{i}\frac{\delta}{\delta Y(t)}(\langle F^I_{W,Z}[X,Y]\rangle_{W,Z})=-2\int_0^t ds\,\alpha^I(t-s)X(s)\times\langle F^I_{W,Z}[X,Y]\rangle_{W,Z}. \qquad (65)
$$

Substituting Eqs. (62)–(65) into Eq. (60), we find the following evolution equation for the noisy density matrix $\hat\rho^S_{\hat\eta_c}(t)$:

$$
\begin{aligned}
i\hbar\frac{\partial}{\partial t}\hat\rho^S_{\hat\eta_c}(t) &= [\hat H^S,\hat\rho^S_{\hat\eta_c}(t)]+\frac{1}{2}[\hat x,[2f(0)\hat x-\hat\eta(t),\hat\rho^S_{\hat\eta_c}(t)]_+] \\
&\quad+\int_0^t ds\,\alpha^I(t-s)\langle[\hat x,\hat U_+(t,s) \\
&\quad\times[\hat x,\hat U_+(s,0)\hat\rho^S(0)\hat U_-(0,s)]_+\hat U_-(s,t)]\rangle_{W,Z}, \qquad (66)
\end{aligned}
$$

where

$$
\hat U_+(s,s')=\hat T\exp\left[-\frac{i}{\hbar}\int_{s'}^{s}du\{\hat H^S+f(0)\hat x^2 -\hat x[W(u)/2+Z(u)+\hat\eta_c(u)]\}\right] \qquad (67)
$$

and

$$
\hat U_-(s',s)=\hat T^{-1}\exp\left[\frac{i}{\hbar}\int_{s'}^{s}du\{\hat H^S+f(0)\hat x^2 -\hat x[-W(u)/2+Z(u)+\hat\eta_c(u)]\}\right], \qquad (68)
$$

where $\hat T$ and $\hat T^{-1}$ are the time ordering and antiordering operators. Equation (66) can also be derived from the relation

$$
\hat\rho^S_{\hat\eta_c;W,Z}(t)=\hat U_+(t,0)\hat\rho^S(0)\hat U_-(0,t), \qquad (69)
$$

by differentiating both sides of Eq. (69) with respect to t and averaging them over all $W(s)$, $Z(s)$ with the weight $P[W,Z]DWDZ$.

Now we define the noisy Heisenberg system operator $\hat O^S_{\hat\eta_c}(t)$ for the Schrödinger system operator $\hat O^S$, as in Sec. IV, by

$$
\mathrm{Tr}_S[\hat O^S\hat\rho^S_{\hat\eta_c}(t)]=\mathrm{Tr}_S[\hat O^S_{\hat\eta_c}(t)\hat\rho^S(0)], \qquad (70)
$$

where $\hat O^S_{\hat\eta_c}(0)=\hat O^S$. By using Eqs. (53) and (69), the left-hand side of Eq. (70) can be rewritten as

$$
\mathrm{Tr}_S[\hat O^S\hat\rho^S_{\hat\eta_c}(t)]=\mathrm{Tr}_S[\langle\hat U_-(0,t)\hat O^S\hat U_+(t,0)\rangle_{W,Z}\hat\rho^S(0)]. \qquad (71)
$$

Here it should be noted that we can interchange the order of the trace over system degrees of freedom and the average with respect to $W(s)$, $Z(s)$. Substituting Eq. (71) into Eq. (70), we obtain

$$
\hat O^S_{\hat\eta_c}(t)=\langle\hat U_-(0,t)\hat O^S\hat U_+(t,0)\rangle_{W,Z}. \qquad (72)
$$

Differentiating both sides of Eq. (70) and using Eqs. (66), (70), and (72), we get

$$
\dot{\hat x}_{\hat\eta_c}(t)=\hat p_{\hat\eta_c}(t)/m, \qquad (73)
$$

$$\dot{\hat{p}}_{\hat{\eta}_c}(t) = -\hat{V}'_{\hat{\eta}_c}(t) - 2f(0)\hat{x}_{\hat{\eta}_c}(t) + \frac{1}{2}[\hat{\eta}(t), \langle \hat{U}_-(0,t)\hat{U}_+(t,0)\rangle_{W,Z}]_+ - \int_0^t ds\, \alpha^I(t-s)\langle \hat{U}_-(0,s) \times [\hat{x}, \hat{U}_-(s,t)\hat{U}_+(t,s)]_+ \hat{U}_+(s,0)\rangle_{W,Z}. \quad (74)$$

Of course, these equations can also be obtained directly by differentiating both sides of Eq. (72) with respect to t and proceeding as we derived Eq. (66) from Eq. (60). The last two terms on the right-hand side of Eq. (74) can be simplified as follows:

$$\frac{1}{2}[\hat{\eta}(t), \langle \hat{U}_-(0,t)\hat{U}_+(t,0)\rangle_{W,Z}]_+ = \hat{\eta}(t), \quad (75)$$

$$\int_0^t ds\, \alpha^I(t-s)\langle \hat{U}_-(0,s)[\hat{x}, \hat{U}_-(s,t)\hat{U}_+(t,s)]_+ \hat{U}_+(s,0)\rangle_{W,Z} = 2\int_0^t ds\, \alpha^I(t-s)\hat{x}_{\hat{\eta}_c}(s). \quad (76)$$

To prove this, we first calculate the derivative with respect to t of $\langle \hat{U}_-(0,t)\hat{U}_+(t,0)\rangle_{W,Z}$. From Eqs. (67) and (68) it can be easily shown that

$$\frac{d}{dt}\langle \hat{U}_-(0,t)\hat{U}_+(t,0)\rangle_{W,Z} = \left\langle \frac{d}{dt}[\hat{U}_-(0,t)\hat{U}_+(t,0)]\right\rangle_{W,Z} = \left\langle \hat{U}_-(0,t)\frac{i}{\hbar}\hat{x}W(t)\hat{U}_+(t,0)\right\rangle_{W,Z}. \quad (77)$$

To further evaluate this we consider the matrix element

$$\langle x_i'| \left\langle \hat{U}_-(0,t)\frac{i}{\hbar}\hat{x}W(t)\hat{U}_+(t,0)\right\rangle_{W,Z} |x_i\rangle. \quad (78)$$

This can be calculated as follows:

$$\begin{aligned} &\langle x_i'| \left\langle \hat{U}_-(0,t)\frac{i}{\hbar}\hat{x}W(t)\hat{U}_+(t,0)\right\rangle_{W,Z} |x_i\rangle \\ &= \left\langle \langle x_i'|\hat{U}_-(0,t)\frac{i}{\hbar}\hat{x}W(t)\hat{U}_+(t,0)|x_i\rangle \right\rangle_{W,Z} \\ &= \left\langle \int_{-\infty}^{\infty} dx \int_{x_i}^{x} Dx_+ \int_{x_i'}^{x} Dx_- \frac{i}{\hbar} x W(t) \times F_{\hat{\eta}_c;W,Z}[x_+,x_-] \exp\left[\frac{i}{\hbar}(A^S[x_+] + A^C[x_+] - A^S[x_-] - A^C[x_-])\right]\right\rangle_{W,Z} \\ &= \frac{i}{\hbar}\int_{-\infty}^{\infty} dx\, x \int_{x_i}^{x} Dx_+ \int_{x_i'}^{x} Dx_- \langle W(t) \times F^I_{W,Z}[x_+,x_-]\rangle_{W,Z} F^R_{\hat{\eta}_c}[x_+,x_-] \exp\left[\frac{i}{\hbar}(A^S[x_+] + A^C[x_+] - A^S[x_-] - A^C[x_-])\right], \end{aligned} \quad (79)$$

where $F^I_{W,Z}[x_+,x_-]$ and $F^R_{\hat{\eta}_c}[x_+,x_-]$ are $F^I_{W,Z}[X,Y]$ and $F^R_{\hat{\eta}_c}[Y]$ in Eq. (52). By using Eqs. (48), (49), and (12), it can be shown that

$$\langle W(t)F^I_{W,Z}[x_+,x_-]\rangle_{W,Z} = \langle W(t)F^I_{W,Z}[X,Y]\rangle_{W,Z} = 0. \quad (80)$$

Substituting this into Eq. (79) we see that

$$\langle x_i'| \left\langle \hat{U}_-(0,t)\frac{i}{\hbar}\hat{x}W(t)\hat{U}_+(t,0)\right\rangle_{W,Z} |x_i\rangle = 0 \quad (81)$$

or

$$\left\langle \hat{U}_-(0,t)\frac{i}{\hbar}\hat{x}W(t)\hat{U}_+(t,0)\right\rangle_{W,Z} = 0. \quad (82)$$

Then, substituting Eq. (82) into Eq. (77), we obtain

$$\frac{d}{dt}\langle \hat{U}_-(0,t)\hat{U}_+(t,0)\rangle_{W,Z} = 0. \quad (83)$$

Hence we have

$$\langle \hat{U}_-(0,t)\hat{U}_+(t,0)\rangle_{W,Z} = \langle \hat{U}_-(0,0)\hat{U}_+(0,0)\rangle_{W,Z} = \langle 1\rangle_{W,Z} = 1. \quad (84)$$

Similarly, we can show that

$$\frac{d}{dt}\langle \hat{U}_-(0,s)[\hat{x}, \hat{U}_-(s,t)\hat{U}_+(t,s)]_+ \hat{U}_+(s,0)\rangle_{W,Z} = 0. \quad (85)$$

Then we have

$$\begin{aligned} &\langle \hat{U}_-(0,s)[\hat{x}, \hat{U}_-(s,t)\hat{U}_+(t,s)]_+ \hat{U}_+(s,0)\rangle_{W,Z} \\ &= \langle \hat{U}_-(0,s)[\hat{x}, \hat{U}_-(s,s)\hat{U}_+(s,s)]_+ \hat{U}_+(s,0)\rangle_{W,Z} \\ &= \langle \hat{U}_-(0,s)[\hat{x}, 1]_+ \hat{U}_+(s,0)\rangle_{W,Z} \\ &= 2\langle \hat{U}_-(0,s)\hat{x}\hat{U}_+(s,0)\rangle_{W,Z} = 2\hat{x}_{\hat{\eta}_c}(s). \end{aligned} \quad (86)$$

Then, from Eqs. (84) and (86), we arrive at Eqs. (75) and (76). Thus, substituting Eqs. (75) and (76) into the right-hand side of Eq. (74) and integrating the last term by parts with respect to s, we get

$$\dot{\hat{x}}_{\hat{\eta}_c}(t)=\hat{p}_{\hat{\eta}_c}(t)/m,$$

$$\dot{\hat{p}}_{\hat{\eta}_c}(t)=-\hat{V}'_{\hat{\eta}_c}(t)-2\int_0^t ds f(t-s)\dot{\hat{x}}_{\hat{\eta}_c}(s)+\hat{\eta}(t)-2f(t)\hat{x}, \tag{87}$$

where we have used Eq. (16). Of course, in the Ohmic case (32), where $f(s)=\gamma\delta(s)$, these equations reduce to Eq. (43). If, again, we recognize that $\hat{V}'_{\hat{\eta}_c}(t)=V'(\hat{x}_{\hat{\eta}_c}(t))$ in Eq. (87), we obtain the same forms as the generalized quantum Langevin equations derived from the Heisenberg equations of motion [12–16]. Then we can again write

$$\hat{x}_{\hat{\eta}_c}(t)=e^{i\hat{H}t/\hbar}\hat{x}e^{-i\hat{H}t/\hbar},$$

$$\hat{p}_{\hat{\eta}_c}(t)=e^{i\hat{H}t/\hbar}\hat{p}e^{-i\hat{H}t/\hbar}. \tag{88}$$

VI. DISCUSSIONS

We have shown how to derive the generalized quantum Langevin equations from the Feynman-Vernon forward-backward path integral, by extending the Kleinert-Shabanov method for the Ohmic environment to the general case. The derivation is based on the supposition of the path integral Fourier transform (45). The functional $P[W,Z]$ in Eq. (45) is formally given by inverting the path integral Fourier transform. However, one does not need to do it. As we have shown, one can get through the work without the explicit form of $P[W,Z]$. In this paper we do not pursue it.

To obtain the same forms as the (generalized) quantum Langevin equations derived from the Heisenberg equations of motion, we have recognized that $\hat{V}'_{\hat{\eta}_c}(t)=V'(\hat{x}_{\hat{\eta}_c}(t))$. However, this relation is nontrivial except for the potential with the form $V(x)=ax^2+bx+c$ with constants a,b,c. In Ref. [9], Kleinert and Shabanov generally define, for any product of system operators $\hat{O}^S=\Pi_{k=1}^n\hat{O}^{(k)}$, a product of noisy Heisenberg system operators by (with our notation) $\hat{O}^S_{\hat{\eta}_c}(t)=\Pi_{k=1}^n\hat{O}^{(k)}_{\hat{\eta}_c}(t)$. We think that the proof of these relations should be given in the future.

Finally we should add the following comments. Equations (43) include the added term $2\gamma\delta(t)\hat{x}$ $[2f(t)\hat{x}$ in Eq. (87)], while not in the original work of Kleinert and Shabanov [9]. This is because their starting point is the influence functional omitting the term corresponding to the second term of the exponent in Eq. (33). In addition, they consider a continuum version of the Hamiltonian (1). Then the quantum noise operator is also given by a continuum version of Eq. (17).

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